

# Finite groups with a certain number of cyclic subgroups

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## Abstract

In this short note, we describe the finite groups  $G$  having  $|G| - 1$  cyclic subgroups. This leads to a nice characterization of the symmetric group  $S_3$ .

In subgroup lattice theory, it is a usual technique to associate to a finite group  $G$  some posets of subgroups of  $G$  (see e.g. [4]). One such poset is the poset of cyclic subgroups of  $G$ , usually denoted by  $C(G)$ . Notice that there are few papers on the connections between  $|C(G)|$  and  $|G|$  ([2, 3] are examples). We also recall the following basic result of group theory.

**Theorem 1.** *A finite group  $G$  is an elementary abelian 2-group if and only if  $|C(G)| = |G|$ .*

Inspired by Theorem 1, we study here the finite groups  $G$  for which

$$(*) \quad |C(G)| = |G| - 1.$$

First, we observe that certain finite groups of small orders, such as  $\mathbb{Z}_3$ ,  $\mathbb{Z}_4$ ,  $S_3$  and  $D_8$ , have this property. Our main theorem proves that in fact these groups exhaust all finite groups  $G$  satisfying  $(*)$ .

**Theorem 2.** *Let  $G$  be a finite group. Then  $|C(G)| = |G| - 1$  if and only if  $G$  is one of the following groups:  $\mathbb{Z}_3$ ,  $\mathbb{Z}_4$ ,  $S_3$  or  $D_8$ .*

*Proof.* Assume that  $G$  satisfies  $(*)$ , let  $n = |G|$  and denote by  $d_1 = 1, d_2, \dots, d_k$  the positive divisors of  $n$ . If  $n_i = |\{H \in C(G) \mid |H| = d_i\}|$ ,  $i = 1, 2, \dots, k$ , then

$$\sum_{i=1}^k n_i \phi(d_i) = n.$$

Since  $|C(G)| = \sum_{i=1}^k n_i = n - 1$ , one obtains

$$\sum_{i=1}^k n_i (\phi(d_i) - 1) = 1,$$

which implies that:

- there exists  $i_0 \in \{1, 2, \dots, k\}$  such that  $n_{i_0} = 1$  and  $\phi(d_{i_0}) = 2$  (i.e.  $d_{i_0} \in \{3, 4, 6\}$ );
- for an  $i \neq i_0$ , we have either  $n_i = 0$  or  $\phi(d_i) = 1$  (i.e.  $d_i \in \{1, 2\}$ ).

We remark that  $d_{i_0}$  cannot be equal to 6 because in this case  $G$  would also have a cyclic subgroup of order 3, a contradiction. We infer that  $G$  contains a unique normal cyclic subgroup of order  $d_{i_0}$ , say  $H$ . Let  $X$  be a subgroup of  $G$  of prime order  $p$ . Then either  $p = 2$  or  $p$  divides  $d_{i_0}$ . Hence by Cauchy's Theorem, either  $d_{i_0} = 4$  and  $G$  is a 2-group, or  $d_{i_0} = 3$  and  $G$  is a  $\{2, 3\}$ -group.

If  $d_{i_0} = 3$  it follows that  $H$  is the unique Sylow 3-subgroup of  $G$ , and consequently  $G = HK$  for some  $K \in \text{Syl}_2(G)$ . The theorem holds if  $G = H$ , so we may assume  $K \neq 1$ . As  $G$  has no cyclic subgroup of order 6, we have  $C_K(H) = 1$ , so  $|K| = |\text{Aut}(H)| = 2$ . Therefore  $G$  is the nonabelian group  $S_3$  of order 6.

Assume next that  $d_{i_0} = 4$ , so that  $G$  is a 2-group. If  $G = H$ , then the theorem holds, so we may assume that there is  $g \in G \setminus H$ . Then, as each member of  $G \setminus H$  is an involution,  $g$  is an involution inverting  $H$  via conjugation. Since  $|G : C_G(H)| \leq |\text{Aut}(H)| = 2$ , we conclude that  $G = H\langle g \rangle$  is the dihedral group of order 8. This completes the proof.  $\square$

The following corollary is an immediate consequence of Theorem 2.

**Corollary 3.**  $S_3$  is the unique finite group  $G$  which is not a  $p$ -group and satisfies  $|C(G)| = |G| - 1$ .

We also remark that a facile proof of Theorem 1 easily follows from the first part of the proof of Theorem 2. Finally, we indicate a natural open problem concerning the above study.

**Open problem.** Describe the finite groups  $G$  satisfying  $|C(G)| = |G| - r$ , where  $2 \leq r \leq |G| - 1$ .

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